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Engineering Notes

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New Quaternion Attitude Estimation Method

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Introduction

THE quaternion parameterization of the rotation group is often preferred for spacecraft attitude dynamics and spacecraft attitude estimation.[1] A quaternion extended Kalman filter has been employed for attitude determination on several spacecraft equipped with rate-integrating gyros, including Landsats 4 and 5, the Gamma Ray Observatory, the Upper Atmosphere Research Satellite, and the Extreme Ultraviolet Explorer.[2,3] However, it is often desirable to have a single-frame attitude estimator, which is an estimator depending only on attitude measurements at a single time.[4] Single-frame attitude estimation is an especially attractive alternative for spacecraft equipped with star trackers that can track several stars simultaneously. The most widely used single-frame attitude estimators are based on an optimality criterion proposed in 1985 by Wahba.[5] She posed the problem of finding the proper orthogonal matrix A that minimizes the nonnegative loss function

$$L(A) \equiv \frac{1}{2} \sum_{i=1}^{n} a_i |\boldsymbol{b}_i - A\boldsymbol{r}_i|^2 \tag{1}$$

where the unit vectors $\boldsymbol{r}_i$ are representations in a reference frame of the directions to some observed objects, the $\boldsymbol{b}_i$ are the unit vector representations of the corresponding observations in the spacecraft body frame, the a_i are positive weights, and n is the number of observations. The motivation for this loss function is that if the vectors are error-free and the true attitude matrix $A_{\rm true}$ is assumed to be the same for all the measurements, then $\boldsymbol{b}_i$ is equal to $A_{\rm true}\boldsymbol{r}_i$ for all i and the loss function is equal to zero for A equal to $A_{\rm true}$. The purpose of this Note is to present a new algorithm for finding the quaternion representation of an attitude matrix that approximately minimizes Wahba's loss function. This approximate solution is shown to agree with the optimal solution for 12 test cases.

New Algorithm

Simple matrix manipulations transform the loss function into

$$L(A) = \lambda_0 - \mathrm{tr}(AB^T) \tag{2}$$

where

$$\lambda_0 \equiv \sum_{i=1}^{n} a_i \tag{3}$$

$$B \equiv \sum_{i=1}^{n} a_i \boldsymbol{b}_i \boldsymbol{r}_i^T \tag{4}$$

tr denotes the trace, and the superscript T denotes the matrix transpose. The weights are often chosen so that $\lambda_0 = 1$, but this is not always the most convenient choice.[6] Wahba's optimization problem has an interesting relation to the matrix norm variously known as the Schur, Frobenius, or Hilbert-Schmidt norm, which is defined for a general real matrix M as the sum of the squares of its elements[7,8]

$$\|M\|^2 \equiv \Sigma M_{ij}^2 = \mathrm{tr}(MM^T) \tag{5}$$

The assumed orthogonality of A and properties of the trace give

$$\|A - B\|^2 = \mathrm{tr}\left[(A - B)(A - B)^T\right] = 3 - 2\,\mathrm{tr}(AB^T) + \|B\|^2 \tag{6}$$

where three results from evaluating the trace of AA^T. The orthogonal matrix A that maximizes $\mathrm{tr}(AB^T)$ minimizes this norm, so Wahba's problem is also equivalent to the problem of finding the proper orthogonal matrix A that is closest to B in the Euclidean norm.[9] It is also related to the problem of finding a "procrustean transformation" of B.[7,8]

Let adj B and det B denote the adjoint and determinant of B, respectively, and let

$$\zeta(\lambda, B) \equiv \tfrac{1}{2}\lambda(\lambda^2 - \|B\|^2) - \det B \tag{7}$$

Then the optimal attitude estimate $A_{\rm opt}$ is given by[6]

$$A(\lambda) = \left[\tfrac{1}{2}(\lambda^2 + \|B\|^2)B + \lambda\,\mathrm{adj}\,B^T - BB^TB\right]/\zeta(\lambda, B) \tag{8}$$

where λ is the largest root of the quartic equation resulting from

$$\lambda = \mathrm{tr}\left[A(\lambda)B^T\right] \tag{9}$$

Iterative and closed-form methods[4,6] are available for finding λ. Other methods for finding the closest orthogonal matrix to a given matrix are mathematically equivalent to solving the quartic equation for λ.[10–14] Many of these methods involve matrix iterations, which are slower than the scalar iterations of Refs. 4 or 6. Approaches that require matrix inverses fail for a singular B matrix, as in the case of two observations. We see in the following that the calculation of λ can be completely avoided with no loss of accuracy.

As first observed by Shuster,[4] λ should be approximately equal to λ_0 since it follows from Eqs. (2) and (9) that

$$L(A_{\rm opt}) = \lambda_0 - \lambda \geq 0 \tag{10}$$

and the loss function should be close to zero for small measurement errors. Thus the attitude estimate resulting from this approximation

$$A_0 \equiv M/\zeta(\lambda_0, B) \tag{11}$$

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with

$$M \equiv \tfrac{1}{2}(\lambda_0^2 + \|B\|^2)B + \lambda_0 \operatorname{adj} B^T - BB^TB \tag{12}$$

should be reasonably accurate. This estimate has the defect of being only approximately orthogonal, though, since $A(\lambda)$ is only orthogonal if λ is a solution of Eq. (9). The attitude matrix can be orthogonalized by extracting a normalized quaternion q from A_0 by a variant of Shepperd's algorithm (note that the sign conventions in this Note agree with Ref. 1 and not with Shepperd).[15] Then the attitude matrix

$$A_{\text{est}} \equiv \begin{bmatrix} q_4^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & q_4^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & q_4^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{13}$$

is guaranteed to be orthogonal.[1]

If i, j, k is a cyclic permutation of 1, 2, 3, the quaternion components obey the relations

$$4\zeta(\lambda_0, B)q_i^2 \approx \zeta(\lambda_0, B) + M_{ii} - M_{jj} - M_{kk} \equiv v_i \tag{14}$$

$$4\zeta(\lambda_0, B)q_iq_j \approx M_{ij} + M_{ji} \equiv v_j \tag{15}$$

$$4\zeta(\lambda_0, B)q_iq_k \approx M_{ik} + M_{ki} \equiv v_k \tag{16}$$

$$4\zeta(\lambda_0, B)q_iq_4 \approx M_{jk} - M_{kj} \equiv v_4 \equiv w_i \tag{17}$$

$$4\zeta(\lambda_0, B)q_jq_4 \approx M_{ki} - M_{ik} \equiv w_j \tag{18}$$

$$4\zeta(\lambda_0, B)q_kq_4 \approx M_{ij} - M_{ji} \equiv w_k \tag{19}$$

and

$$4\zeta(\lambda_0, B)q_4^2 \approx \zeta(\lambda_0, B) + M_{ii} + M_{jj} + M_{kk} \equiv w_4 \tag{20}$$

The approximations in Eqs. (14-20) are as accurate as the approximation $A_0 \approx A_{\text{est}}$. The essence of Shepperd's algorithm is to avoid computing the square root of a small number or the ratio of two small numbers. Thus we let i be the index of the largest diagonal element of M and define the quaternion components for $l = 1, \ldots, 4$ by

$$q_l = v_l/|v| \qquad \text{for} \quad M_{jj} + M_{kk} < 0 \tag{21}$$

or

$$q_l = w_l/|w| \qquad \text{for} \quad M_{jj} + M_{kk} \geq 0 \tag{22}$$

where $|v|$ and $|w|$ denote the Euclidean norms of the four-vectors v and w, respectively. Equations (3), (4), (7), (12), and (14-22) define the new algorithm completely.

Tests

The new method was compared with the optimal matrix method (FOAM)[6] for 12 test cases. Both methods were implemented in G_FLOATING FORTRAN on a DEC VAX 8830. Each test case was specified by a set of measurement vectors $\boldsymbol{r}_i$ and measurement standard deviations σ_i. The observation vectors were computed as

$$\boldsymbol{b}_i = A_{\text{true}}\boldsymbol{r}_i + \boldsymbol{n}_i \tag{23}$$

where $\boldsymbol{n}_i$ is a vector of measurement errors and

$$A_{\text{true}} = \begin{bmatrix} 0.352 & 0.864 & 0.360 \\ -0.864 & 0.152 & 0.480 \\ 0.360 & -0.480 & 0.800 \end{bmatrix} \tag{24}$$

which has all nonzero matrix elements with exact decimal representations and is otherwise arbitrary. The tests were run both with $\boldsymbol{n}_i = \boldsymbol{0}$ and with measurement errors simulated by zero-mean Gaussian white noise on the components of $\boldsymbol{n}_i$. The specified measurement standard deviations in each case were used to compute the level of simulated measurement errors and also the measurement weights

$$a_i = \sigma_i^{-2} \tag{25}$$

The 12 test cases were specified as follows:

Case 1 used the three reference vectors

$$r_1 = [1, 0, 0]^T, \qquad r_2 = [0, 1, 0]^T, \qquad r_3 = [0, 0, 1]^T \tag{26}$$

with measurement standard deviations $\sigma_1 = \sigma_2 = \sigma_3 = 10^{-6}$ rad. This reference vector set models three fine sensors with orthogonal boresights along the body axes.

Case 2 used the two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from the previous set with $\sigma_1 = \sigma_2 = 10^{-6}$ rad.

Case 3 was the same as case 1 but with $\sigma_1 = \sigma_2 = \sigma_3 = 0.01$ rad, modeling three orthogonal coarse sensors.

Case 4 was the same as case 2 but with $\sigma_1 = \sigma_2 = 0.01$ rad.

Case 5 used the two reference vectors

$$r_1 = [0.6, 0.8, 0]^T, \qquad r_2 = [0.8, -0.6, 0]^T \tag{27}$$

with $\sigma_1 = 10^{-6}$ rad and $\sigma_2 = 0.01$ rad. This models one fine and one coarse sensor with orthogonal boresights not along the spacecraft body axes.

Case 6 used the three reference vectors

$$r_1 = [1, 0, 0]^T, \quad r_2 = [1, 0.01, 0]^T, \quad r_3 = [1, 0, 0.01]^T \tag{28}$$

with $\sigma_1 = \sigma_2 = \sigma_3 = 10^{-6}$ rad. This reference vector set models three star measurements in a single star sensor with a small field-of-view.

Case 7 used the two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from Eq. (28) with $\sigma_1 = \sigma_2 = 10^{-6}$ rad.

Case 8 was the same as case 6 but with $\sigma_1 = \sigma_2 = \sigma_3 = 0.01$ rad, modeling a star sensor with large errors, to stress the algorithms.

Case 9 was the same as case 7 but with $\sigma_1 = \sigma_2 = 0.01$ rad.

Case 10 used the three reference vectors

$$r_1 = [1, 0, 0]^T, \quad r_2 = [0.96, 0.28, 0]^T, \quad r_3 = [0.96, 0, 0.28]^T \tag{29}$$

with $\sigma_1 = 10^{-6}$ rad and $\sigma_2 = \sigma_3 = 0.01$ rad. This models one fine sensor with its boresight along the body x axis and two less accurate reference vectors 16.26 deg off this axis.

Case 11 used the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from Eq. (29) with $\sigma_1 = 10^{-6}$ rad and $\sigma_2 = 0.01$ rad.

Case 12 was the same as case 11 but with $\sigma_1 = 0.01$ rad and $\sigma_2 = 10^{-6}$ rad. This models the case that the boresight of the fine sensor is 16.26 deg off the body x axis.

The accuracy measure of most interest in applications is the estimation error, which is defined as the rotation angle between the true and estimated attitudes, and is computed by[6]

$$\phi_{\text{err}}(A) = 2\sin^{-1}\left(\|A - A_{\text{true}}\|/\sqrt{8}\right) \tag{30}$$

The estimation error and the loss function for the 12 test cases, computed with simulated measurement errors, are presented in Table 1.

The results for the two methods are virtually identical except in test cases 8 and 9, which were deliberately chosen to stress the algorithms. The new method gives a smaller error than the optimal method in one of these two cases, where the measurement errors are as large as the angular separations between the

Table 1 Estimation error and loss function

	Matrix method		New method	
Case	$\phi_{err}(A_{opt})$, rad	$L(A_{opt})$	$\phi_{err}(A_{est})$, rad	$L(A_{est})$
1	1.23×10^{-6}	1.81	1.23×10^{-6}	1.81
2	1.79×10^{-6}	1.15	1.79×10^{-6}	1.15
3	1.25×10^{-2}	1.86	1.24×10^{-2}	1.86
4	1.81×10^{-2}	1.18	1.80×10^{-2}	1.18
5	1.21×10^{-2}	0.07	1.21×10^{-2}	0.07
6	3.10×10^{-5}	2.19	3.10×10^{-5}	2.19
7	3.94×10^{-5}	1.70	3.94×10^{-5}	1.70
8	0.235	2.26	0.192	2.42
9	0.105	1.78	0.205	1.86
10	2.17×10^{-2}	2.13	2.11×10^{-2}	2.13
11	4.22×10^{-2}	0.13	4.21×10^{-2}	0.13
12	2.74×10^{-2}	2.34	2.84×10^{-2}	2.34

measurements, and a smaller error in the other case. The estimation errors of the two methods are the same in cases with $\boldsymbol{n}_i = \boldsymbol{0}$, but the new method gives a more accurately orthogonal attitude matrix. It is very nice that no significant loss of accuracy results from omitting the optimization of λ, since this optimization can be time consuming and very sensitive to numerical inaccuracies.[6]

The computational speed of the new method was also compared with FOAM and with QUEST,[4] which are the fastest known methods for minimizing Wahba's loss function. The measured CPU times were computed for sets of 2 to 12 observations similar to those given by Eq. (28). They consist of a part that is independent of the number of observations and a part proportional to the number of observations

$$t_{new} = 0.28 + 0.07n \text{ msec} \tag{31}$$

$$t_{FOAM} = 0.26 + 0.07n \text{ msec for } \sigma_i = 10^{-6}\text{ rad} \tag{32}$$

$$t_{FOAM} = 0.32 + 0.07n \text{ msec for } \sigma_i = 0.01\text{ rad} \tag{33}$$

$$t_{QUEST} = 0.24 + 0.09n \text{ msec for } \sigma_i = 10^{-6}\text{ rad} \tag{34}$$

$$t_{QUEST} = 0.30 + 0.09n \text{ msec for } \sigma_i = 0.01\text{ rad} \tag{35}$$

The n-dependent time is the time required to normalize the input vectors and form the B matrix. The n-independent time is the time required to perform all other computations, including computation of the attitude error angle covariance matrix.[4,6] The time required for the iterative optimization of λ in FOAM or QUEST is seen to depend on the level of measurement errors. The new method, on the other hand, is executed in a fixed amount of time, since a fixed number of mathematical operations is performed. The absolute execution times are not of great significance; only the relative times are interesting. In any case, the time required for any of the three methods appears to be quite modest in comparison with other computations performed in spacecraft attitude determination.

Conclusions

A new quaternion estimation algorithm that finds an approximate minimum of Wahba's loss function has been shown to provide attitude estimates as accurate as the optimal estimates. The new algorithm is also as computationally efficient and robust as existing algorithms in the test cases examined. The attitude matrix and the quaternion are inherently nonsingular, and any potential problems with special cases like 180-deg rotations are avoided by using Shepperd's algorithm to extract the quaternion from the attitude matrix. A significant advantage of the new method for onboard applications is that it involves no iterative processes that require an uncertain amount of execution time and present the possibility of divergence.

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Rendezvous Guidance with Proportional Navigation

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I. Introduction

PROPORTIONAL navigation has been widely used as the guidance scheme in the homing phase of flight for most missile systems.[1–13] Under these guidance schemes, the component of relative velocity in the direction normal to the line-of-sight (LOS) between an interceptor and its target is always driven to zero during intercept course, and the component of relative velocity along the LOS is not required to approach zero for effective intercept of target. However, for the rendezvous problem of two space vehicles, the relative velocity must be driven to zero when the two vehicles meet. For this reason the commanded acceleration of the active vehicle must be applied in both the direction normal to LOS and the direction along LOS simultaneously to reduce the relative velocity to zero as the two vehicles approach each other. Besides rendezvous guidance laws studied before,[14–18] a new rendezvous guidance scheme for use on the terminal phase in an approach

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